Hyperbola Notes (section 8.3) Precalculus

take note

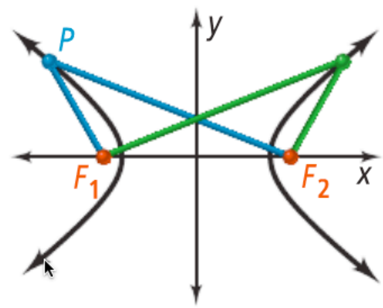
Key Concept Hyperbola

A **hyperbola** is the set of points P in a plane such that the absolute value of the difference between the distances from P to two fixed points F_1 and F_2 is a constant k.

$$|PF_1 - PF_2| = k, \text{ where } k < F_1F_2$$

Each fixed point F is a **focus of the hyperbola**.

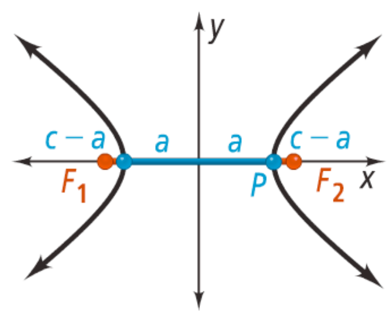
Since F_1 and F_2 are the foci of the hyperbola, the long and short segments in each of the 2 colored paths differ in length by k.



A hyperbola consists of two smooth branches. The turning point of each branch is a **vertex** of the hyperbola. The segment connecting the two vertices is the **transverse axis**, which lies on the **axis of symmetry**. The two foci also lie on the axis of symmetry. The **center of the hyperbola** is the midpoint between the two vertices, which also is the midpoint between the two foci.

Just as for an ellipse, if the foci are $(\pm c, 0)$, the distance between the two foci is $2c$. If the vertices are $(\pm a, 0)$, the distance between the vertices is $2a$.

Since vertex P is on the hyperbola, it must satisfy the equation $|PF_1 - PF_2| = k$, but you can also see that



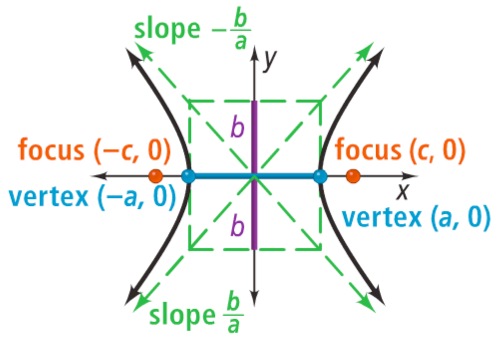
$$\begin{aligned} |PF_1 - PF_2| &= |[2a + (c - a)] - (c - a)| \\ &= |2a + c - a - c + a| \\ &= |2a| = 2a \end{aligned}$$

Therefore, $k = 2a$.

In a standard hyperbola, c is related to a and b by the equation $c^2 = a^2 + b^2$. The length of the **conjugate axis** is $2b$. The transverse and conjugate axes determine a rectangle that lies between the vertices, and the diagonals of that central rectangle determine the asymptotes of the hyperbola. Recall that an asymptote is a line that a graph approaches. The branches of the hyperbola will approach the asymptotes.

Key Concept Properties of Hyperbolas with Center (0, 0)

Horizontal Hyperbola



Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

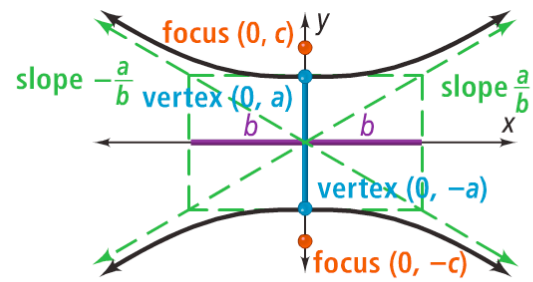
Transverse axis: Horizontal

Vertices: $(\pm a, 0)$

Foci: $(\pm c, 0)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{b}{a}x$

Vertical Hyperbola



Equation: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Transverse axis: Vertical

Vertices: $(0, \pm a)$

Foci: $(0, \pm c)$, where $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{a}{b}x$

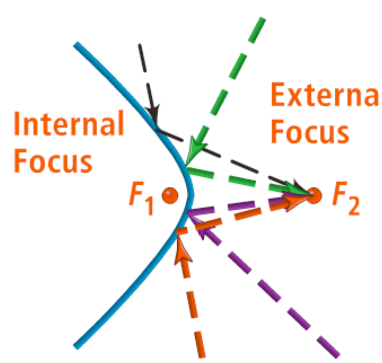
How to sketch a hyperbola:

1. Sketch the line segments at ***x = a, x = -a, y = b,*** and ***y = -b*** to create a rectangle

2. Sketch the asymptotes by extending the rectangle's diagonals.

3. Use the rectangle and the extended diagonals to guide your drawing.

The *reflection property of a hyperbola* is important in optics. As with an ellipse, the reflection property of a hyperbola involves both foci, but only one branch reflects. Any ray on the *external side* of a branch directed at its internal focus will reflect off the branch toward the *external focus*.



Horizontal Hyperbola	**Center (h, k)**
Standard-Form Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Vertices	$(h \pm a, k)$
Foci	$(h \pm c, k)$
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$
a, b, c relationship	$c^2 = a^2 + b^2$

Vertical Hyperbola	**Center (h, k)**
Standard-Form Equation	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Vertices	$(h, k \pm a)$
Foci	$(h, k \pm c)$
Asymptotes	$y - k = \pm\frac{a}{b}(x - h)$
a, b, c relationship	$c^2 = a^2 + b^2$