PART 1
From Sets to Functions

Section 1: Sets and Interval Notation

Introduction

Set concepts offer the means for understanding many different aspects of mathematics and its applications to other branches of learning. For this reason the study of sets receives its fair share of attention at many levels of mathematical instruction.

Because of the importance of sets it should not come as any surprise to find the use of sets in explaining many phenomena, regardless of your choice of study. Those who chose science or technology fields, as well as those who have a simple need to apply mathematics in non-scientific fields, will at some point find themselves confronted with using sets.

Several of the concepts of sets, such as the ones we will study here, are simple and easily understood. However set theory is a very broad and far reaching branch of mathematics. Entire courses are built around their study. Conversely, we have a much smaller focus regarding their study.

Our study of sets will focus on such concepts as writing sets and set operations. As a refresher to what you may already know, we will begin with a formal definition of a set.

DEFINITION OF A SET

A **set** is a well defined collection of objects. The objects in the set are called the *elements* of the set.

Our focus here will be on a few of the more commonly used methods for indicating sets: (1) the listing method, (2) set-builder notation, and (3) finite and infinite sets. The next few Lessons will illustrate each of these beginning with the listing method.

Lesson 1 – The Listing Method

Listing the individual elements of a set within a set of braces is called the ***listing method***, or ***roster method***. Traditionally, capital letters are used to identify or name a given set and braces are used to enclose the elements.

For example, $A = \{1, 2, 3, 4\}$ may be used as an illustration of the listing method for the set of natural numbers less than five. The choice of using the letter A to name this set was totally arbitrary, meaning any capital letter could have been used.

Let's take the time here to formally define some important number sets using the listing method and introducing another concept used in set theory – *ellipses*.

DEFINITIONS OF IMPORTANT NUMBER SETS

The **natural numbers** are the set of counting numbers.

$$N = \{1, 2, 3, 4, 5, 6, . . . \}.$$

The **whole numbers** are the natural number plus the number 0.

$$W = \{0, 1, 2, 3, 4, 5, 6, . . . \}.$$

The **integers** are the positive and negative natural numbers plus 0.

$$I = \{ . . . , -3, -2, -1, 0, 1, 2, 3, . . . \}.$$

The **rational numbers** are any numbers which can be written as a fraction.

$$Q = \{p/q, \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}.$$

Irrational numbers are numbers that have a non-repeating, non-terminating decimal representation.

The **real numbers** consists of both the sets of rational and irrational numbers.

NOTE:

Note examples of irrational numbers would include numbers like π and the square root of 2.

We will reserve the use of the italicized capital letter N throughout this book to represent the set of natural numbers, and the letter W to represent the whole numbers. The three dots after the number 6 are called **ellipses** and indicate that there are other elements of these set that are not listed. Ellipses should only be used only after it is clear that a pattern has been established and that the other elements of the set can easily be determined by continuing in the same manner. Ellipses can also be used to indicate there are elements not listed up to some last element.

For example, the listing of $B = \{2, 4, 6, 8, 10, . . . , 50\}$ shows that set B is the set of all even natural numbers from 2 up to 50.

Example 1.1.1

Write each of the following using the listing method.

(a) Set A is the set of whole numbers less than 10.
(b) Set B is the set of odd natural numbers less than or equal to 75.
(c) Set C is the set of whole number multiples of 5.
(d) Set D is the set of natural number between 6 and 10.
(e) Set E is the set of natural numbers between 6 and 10, inclusive.

Solutions

. . . to part (a) $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

. . . to part (b) $B = \{1, 3, 5, 7, 9, \ldots, 75\}$

. . . to part (c) $C = \{0, 5, 10, 15, 20, 25, \ldots\}$

. . . to part (d) $D = \{7, 8, 9\}$

. . . to part (e) $E = \{6, 7, 8, 9, 10\}$

Of course, all parts of the Example above are important. But notice in parts (d) and (e) how the solutions differ, and more importantly, *why* they differ. In part (d) the word *between* literally means that, the natural numbers *between* 6 and 10. While in part (e) the use of the word *inclusive* at the end means to include the numbers 6 and 10 in your listing of the elements of this set.

Lesson 2 – Set-Builder Notation

HISTORICAL NOTE:

The symbol $\in$*, which means "is an element of" was first used by Peano in 1889; it is an abbreviation of the Greek word εστι, meaning "is".*

A second method to illustrate a set is ***set-builder notation***. This is also known as *set-generator notation*. Set-builder notation is used in many other mathematics courses like liberal arts math and calculus. The following will illustrate its form.

A	$=$	$\{$	x	$\mid$	*conditions* $\}$
↑	↑	↑	↑	↑	↑
Set A	is	the set of	all x's	such that	the condition(s) x must meet in order to be a member of the set

As a concrete illustration, consider the set $E = \{x \mid x \in N \text{ and } x \leq 10\}$.

This statement is read "Set E is the set of all x's such that x is an element of the natural numbers and x is less than or equal to 10".

The conditions that x must meet in order to be a member of this set is that x must be a natural number and x must be less than 10. If employing the listing method for this set E we would write $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Example 1.1.2

Write each of the following sets using set-builder notation. (There may be more than one way to write the solution.)

(a) Set A is the set of whole numbers between 6 and 15.
(b) $B = \{2, 4, 6, 8, 10, 12, \ldots\}$
(c) $C = \{2, 4, 8, 16, 32, \ldots\}$

Solutions

. . . to part (a) $A = \{x \mid x \in W \ \textit{and} \ 6 < x < 15\}$

. . . to part (b) $B = \{x \mid x \in N \ \textit{and} \ x \ \textit{is even}\}$

. . . to part (c) $C = \{x \mid x \ \textit{is a natural number power of} \ 2\}$

In part (c) above you are being shown what we mean by "a natural number power of 2". This is $2^1, 2^2, 2^3, 2^4, \ldots$ The number (base) 2 is being raised to consecutive powers of natural numbers beginning with 1. Can you determine what the first element is if we used whole number powers of 2?

Example 1.1.3

Write each of the following sets using the listing method.

(a) $A = \{x \mid x \in N \ \textit{and} \ x \ \textit{is odd}\}$
(b) $B = \{x \mid x \in W \ \textit{and} \ 3 \leq x < 10\}$

Solutions

. . . to part (a) $A = \{1, 3, 5, 7, 9, \ldots\}$

. . . to part (b) $B = \{3, 4, 5, 6, 7, 8, 9\}$

NOTE:

Just because a set contains the ellipses (3 dots) does not mean it is an infinite set. Consider the set {2, 4, 6, 8, . . . , 50}. This is the set of all even numbers from 2 up to 50.

Lesson 3 – Finite and Infinite Sets

A set such as A = $\{1, 2, 3, 4, 5, \ldots\}$ is considered an **infinite** set because it continues without end. While a set such as $B = \{3, 6, 9, 12\}$ is a **finite** set because we can list all of its elements.

There is actually a more advanced way to determine whether or not a set is finite or infinite, but for our purposes an intuitive understanding will suffice for now.

Example 1.1.4

Determine whether the given sets are finite or infinite.

(a) $A = \{2, 4, 8, 16, 32, \ldots, 1024\}$
(b) $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
(c) $C = \{x \mid x \in W \text{ and } x < 100\}$
(d) $D = \{x \mid x \in N \text{ and } x > 100\}$
(e) $E = \{x \mid x \in N \text{ and } x \leq 0\}$

Solutions

. . . to part (a) Finite. These are the first ten natural number powers of 2.

. . . to part (b) Infinite.

. . . to part (c) Finite. This is the set $C = \{0, 1, 2, 3, 4, \ldots, 99\}$.

. . . to part (d) Infinite, since $D = \{101, 102, 103, 104, \ldots\}$

RECALL:

*The **empty set** is a set that contains no elements. It is also called the null set.*

. . . to part (e) Finite. This is an example of an ***empty set*** since there are no natural numbers that are less than or equal to 0. We can write $E = \{\ \}$ or Ø.

Lesson 4 – Set Operations – Union and Intersection

There are two set operations we are to learn in this course: the **union** operation and the **intersection** operation. These are two relatively easy operations to perform.

We begin by defining the union of two sets.

DEFINITION OF THE UNION OPERATION

The **union** of two sets A and B, written as $A \cup B$, is the set of all elements belonging to either of the two sets. In symbols we write,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

NOTE:

Note to associate the word or with the union operation.

We can form the union of two sets by taking all of the elements from each and combining them into one larger set.

To illustrate, if $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 4, 5, 6, 7\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Note how it is not necessary to write the repeated elements of 5 and 7. Repetition elements need to only be listed once.

In this illustration above the two sets A and B have elements in common, namely 5 and 7. These common elements form another set we call the **intersection**.

DEFINITION OF THE INTERSECTION OPERATION

The **intersection** of two sets A and B, written as $A \cap B$, is the set of all elements common to both sets. In symbols we write,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

NOTE:

Note to associate the word and with the intersection operation.

To illustrate, if $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 4, 5, 6, 7\}$, then

$$A \cap B = \{5, 7\}.$$

Example 1.1.5

Write the union and intersection for each of the given pairs of sets.

(a) $A = \{2, 3, 5, 8, 9\}$ and $B = \{0, 2, 4, 6, 8, 9\}$
(b) $C = \{1, 3, 5, 7, 9\}$ and $D = \{3, 4, 5, 6, 7\}$
(c) $E = \{1, 5, 7, 8\}$ and $F = \{0, 2, 3, 4\}$

Solutions

. . . to part (a) $A \cup B = \{0, 2, 3, 4, 5, 6, 8, 9\}$
$A \cap B = \{2, 8, 9\}$

. . . to part (b) $C \cup D = \{1, 3, 4, 5, 6, 7, 9\}$
$C \cap D = \{3, 5, 7\}$

. . . to part (c) $E \cup F = \{0, 1, 2, 3, 4, 5, 7, 8\}$
$E \cap F = \{\ \}$ or Ø

Lesson 5 – Interval Notation

Sets can also be written using interval notation. To illustrate, the interval notation $(-2, 3]$ indicates the set of all real numbers greater than -2 and less than or equal to 3. The numbers -2 and 3 are called **endpoints**. The graphical representation of this interval is shown below in Figure 1.1.1.

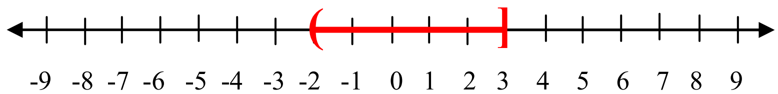


Figure 1.1.1 The interval of $(-2, 3]$.

In the interval notation and in the graphical representation of Figure 1.1.1, the parenthesis indicates the number -2 is not to be included in the set. The bracket indicates that the number 3 is to be included in the set.

NOTE:

Note the different names for intervals. Be sure you understand them and how to represent each of them graphically.

An interval is said to be **closed** if it includes both endpoints. To illustrate, the interval of $[-2, 3]$ is a closed interval. It is the set of all real numbers greater than or equal to -2 and less than or equal to 3. Figure 1.1.2 illustrates.

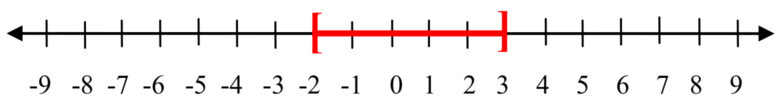


Figure 1.1.2 The closed interval of $[-2, 3]$.

An interval is said to be **open** if it does not include either endpoint. To illustrate, the interval of $(-2, 3)$ is an open interval. It is the set of all real numbers greater than -2 and less than 3.

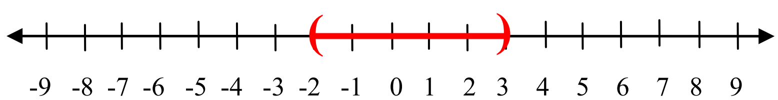


Figure 1.1.3 The open interval of $(-2, 3)$.

An interval is said to be **half-open** if it includes just one endpoint. To illustrate, the interval $(-2, 3]$ is half-open and indicates the set of all real numbers greater than -2 and less than or equal to 3.

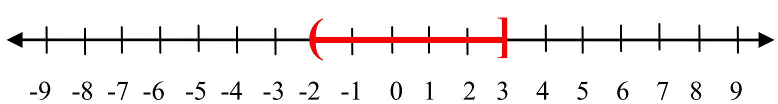


Figure 1.1.4 The half-open interval of $(-2, 3]$.

Set-builder notation and interval notation can be linked as illustrated in Table 1.1.1.

Set-builder notation	Interval notation	Type of interval
$\{x \mid -2 \le x \le 3\}$	$[-2, 3]$	Closed interval
$\{x \mid -2 < x < 3\}$	$(-2, 3)$	Open interval
$\{x \mid -2 < x \le 3\}$	$(-2, 3]$	Half-open interval

Table 1.1.1

NOTE:

Note the infinity symbol is not a number. It is simply a symbol that indicates a set being infinite, or never-ending.

If a set is to extend indefinitely in either, or both, directions, then we can use the infinity symbol ∞ or negative infinity symbol $-\infty$. To illustrate, the interval notation of $[2, \infty)$ indicates the set of all real numbers greater than or equal to 2, while the interval notation of $(-\infty, 2)$ indicates the set of all real numbers less than 2.

In the two examples of the interval notation above notice how the parenthesis is used in conjunction with the infinity symbol. Because infinity is not a real number a parenthesis is always used: a left parenthesis with $-\infty$, and a right parenthesis with ∞. The figures below illustrate the graphs of each of these two intervals.

NOTE:

Note in the graphical representation of these two sets how the parenthesis is <u>*not*</u> *included on the graph itself. Why?*

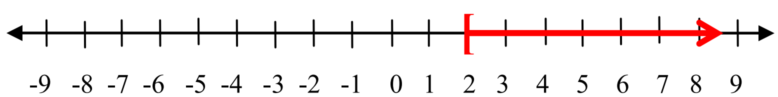


Figure 1.1.5 The interval of $[2, \infty)$.

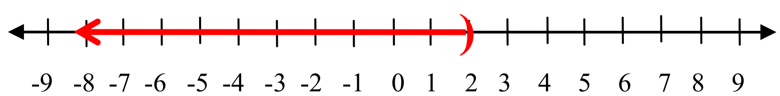


Figure 1.1.6 The interval of $(-\infty, 2)$.

Our next example of this section blends much of what we have been learning.

Example 1.1.6

Write the interval notation and illustrate graphically each of the given sets. Also indicate the type of interval it is.

(a) $\{x \mid -8 \le x \le 3\}$
(b) $\{x \mid x > -2\}$
(c) $\{x \mid x \le 0\} \cup \{x \mid x > 3\}$
(d) $\{x \mid x < 4\} \cap \{x \mid x > -5\}$
(e) $\{x \mid -\infty < x < \infty\}$

Solutions

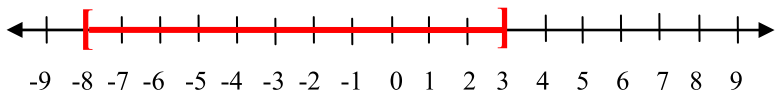
. . . to part (a)

Set-builder notation: $\{x | -8 \le x \le 3\}$

Interval notation: $[-8, 3]$

Type of interval: Closed interval

Graphical representation



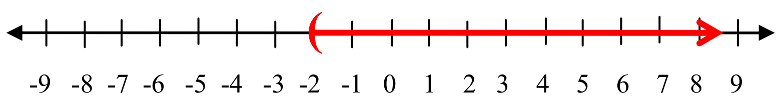
. . . to part (b)

Set-builder notation: $\{x | x > -2\}$

Interval notation: $(-2, \infty)$

Type of interval: Open interval

Graphical representation



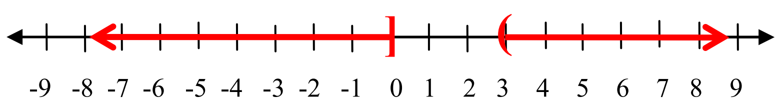
. . . to part (c)

Set-builder notation: $\{x \mid x \le 0 \text{ or } x > 3\}$

Interval notation: $(-\infty, 0]$ or $(3, \infty)$

Type of intervals: Half-open interval and open interval

Graphical representation



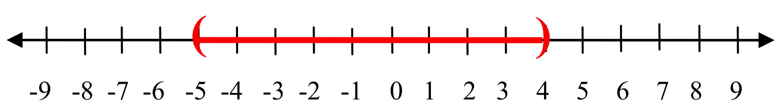
. . . to part (d)

Set-builder notation: $\{x \mid x < 4 \text{ and } x > -5\}$

Interval notation: $(-5, 4)$

Type of interval: Open interval

Graphical representation



NOTE:

Note when using the infinity symbol we always use the strict inequality sign of < (less than). You should never write $-\infty \le x$ or $x \le \infty$.

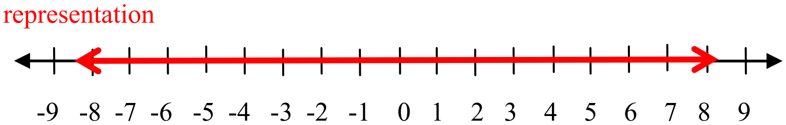
. . . to part (e)

Set-builder notation: $\{x | -\infty < x < \infty\}$

Interval notation: $(-\infty, \infty)$

Type of interval: Open interval

Graphical representation



Exercises for §1.1

In exercises 1 through 8 use the listing method to write the given sets. Use ellipses when necessary.

1. The integers between −6 and 4.
2. The integers between −3 and 10.
3. The odd natural numbers less than 16.
4. The even natural numbers greater than 20.
5. Positive natural number multiples of 5 that are less than or equal to 50.
6. $A = \{x \mid x \in N \text{ } and \text{ } 3 \le x < 10\}$.
7. $B = \{x \mid x \in W \text{ } and \text{ } x < 7\}$.
8. The whole number powers of 2.

In exercises 9 through 16 use the set-builder notation to write the given sets.

9. The integers less than −5.
10. The integers greater than or equal to −5.
11. The even natural numbers less than 20.
12. The odd natural numbers greater than 0.
13. The real numbers between 1 and 2.
14. The real numbers between −3/2 and 7/2.
15. The real numbers between 0 and 1, inclusive.
16. The whole number powers of 3.

In exercises 17 through 24 determine whether the given sets are finite or infinite.

17. The integers less than −5.
18. $A = \{1, 2, 4, 8, 16, 32, \ldots, 2048\}$
19. The even natural numbers less than 20.
20. $B = \{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\}$
21. $C = \{x \mid x \in N \text{ } and \text{ } x > 25\}$
22. The real numbers between 0 and 1.
23. $D = \{x \mid x \in W \text{ } and \text{ } x < 0\}$
24. $E = \{1, -1, 2, -2, 3, -3, 4, -4, 5, -5\}$

In exercises 25 through 32 find $A \cup B$.

25. $A = \{1, 3, 5, 7\}$; $B = \{0, 2, 3, 4, 8, 9\}$
26. $A = \{0, 2, 8, 9\}$; $B = \{0, 2, 4, 6, 7, 10\}$
27. $A = \{1, 3, 5, 7, \ldots\}$; $B = \{0, 2, 4, 6, \ldots\}$
28. $A = \{1, 5, 8,\}$; $B = \{7, 6, 5\}$
29. $A = \{x \mid x \in N \text{ } and \text{ } x < 25\}$; $B = \{x \mid x \in N \text{ } and \text{ } x \ge 25\}$
30. $A = \{x \mid x \in W \text{ } and \text{ } x \le 5\}$; $B = \{x \mid x \in N \text{ } and \text{ } x < 7\}$
31. $A = \{2, 9, 4, 7, 6, 5\}$; $B = \{0, 2, 3, 4, 5\}$
32. $A = \{5, 10, 15, 20, \ldots\}$; $B = \{10, 20, 30\}$

In exercises 33 through 40 find $A \cap B$.

33. $A = \{1, 3, 5, 7\}$; $B = \{0, 2, 3, 4, 8, 9\}$

34. $A = \{0, 2, 8, 9\}$; $B = \{0, 2, 4, 6, 7, 10\}$

35. $A = \{1, 3, 5, 7, \ldots\}$; $B = \{0, 2, 4, 6, \ldots\}$

36. $A = \{1, 5, 8,\}$; $B = \{8, 7, 6, 5\}$

37. $A = \{x \mid x \in N \text{ and } x < 25\}$;
$B = \{x \mid x \in N \text{ and } x \geq 25\}$

38. $A = \{x \mid x \in W \text{ and } x \leq 5\}$;
$B = \{x \mid x \in N \text{ and } x < 7\}$

39. $A = \{2, 9, 4, 7, 6, 5\}$; $B = \{0, 2, 3, 4, 5\}$

40. $A = \{5, 10, 15, 20, \ldots\}$; $B = \{10, 20, 30\}$

In exercises 41 through 48 write each of the given intervals in set-builder notation and then graph.

41. $(-3, 6)$

42. $[-7, 7]$

43. $[-5, 0)$

44. $(2, 8]$

45. $[1, 8]$

46. $(4, \infty)$

47. $[-4, \infty)$

48. $[1, 9]$

In exercises 49 through 56 write each of the given sets of real numbers in interval notation.

49. $\{x|-10 \leq x \leq 5\}$

50. $\{x|x > -1\}$

51. $\{x|-6 < x \leq 4\}$

52. $\{x|0 < x < 3\}$

53. $\{x|x \leq -1\}$

54. $\{x|-\infty < x \leq 2\}$

55. $\{x|-2 < x < \infty\}$

56. $\{x|-7 \leq x < -4\}$

In exercises 57 through 68 graph each of the given sets of real numbers.

57. $\{x|-8 \leq x \leq 3\}$

58. $\{x|x > -4\}$

59. $\{x|-1 < x \leq 4\}$

60. $\{x|0 < x < 5\}$

61. $\{x|x \leq -2\}$

62. $\{x|-\infty < x \leq 2\}$

63. $\{x|x > 0\} \cup \{x|x \leq -3\}$

64. $\{x|x \leq -2\} \cup \{x|x \geq 1\}$

65. $\{x|x > -6\} \cap \{x|x \leq 1\}$

66. $\{x|x > 1\} \cap \{x|x \geq -5\}$

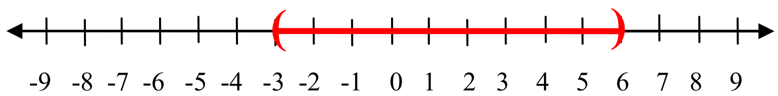
67. $\{x|x < -3\} \cap \{x|x > 4\}$

68. $\{x|x \geq -2\} \cup \{x|x \geq 1\}$

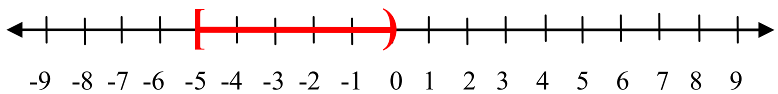
Answers for §1.1 Exercises

1. $\{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$
2. $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
3. $\{1, 3, 5, 7, 9, 11, 13, 15\}$
4. $\{22, 24, 26, 28, 30, \ldots\}$
5. $\{5, 10, 15, 20, \ldots, 50\}$
6. $A = \{3, 4, 5, 6, 7, 8, 9\}$
7. $B = \{0, 1, 2, 3, 4, 5, 6\}$
8. $\{1, 2, 4, 8, 16, 32, \ldots\}$
9. $\{x \mid x \in I \text{ and } x < -5\}$
10. $\{x \mid x \in I \text{ and } x \geq -5\}$
11. $\{x|x$ is an even natural number and $x < 20\}$
12. $\{x|x$ is an odd natural number and $x > 0\}$
13. $\{x|x$ is a real number and $1 < x < 2\}$
14. $\{x|x$ is a real number and $-3/2 < x < 7/2\}$
15. $\{x|x$ is a real number and $0 \leq x \leq 1\}$
16. $\{x|x$ is a whole number power of $3\}$
17. Infinite
18. Finite
19. Finite
20. Infinite
21. Infinite
22. Infinite
23. Finite
24. Finite
25. $A \cup B = \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$
26. $A \cup B = \{0, 2, 4, 6, 7, 8, 9, 10\}$
27. $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$
28. $A \cup B = \{1, 5, 6, 7, 8\}$
29. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$
30. $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$
31. $A \cup B = \{0, 2, 3, 4, 5, 6, 7, 9\}$
32. $A \cup B = \{5, 10, 15, 20, \ldots\}$
33. $A \cap B = \{3\}$
34. $A \cap B = \{2, 4\}$
35. $A \cap B = \emptyset$ or $\{\ \}$
36. $A \cap B = \{5, 8\}$
37. $A \cap B = \emptyset$ or $\{\ \}$
38. $A \cap B = \{1, 2, 3, 4, 5\}$
39. $A \cap B = \{2, 4, 5\}$
40. $A \cap B = \{10, 20, 30\}$

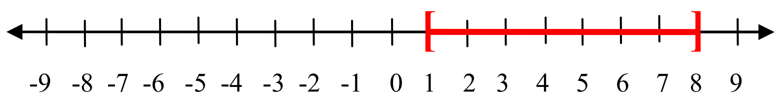
41. $\{x | -3 < x < 6\}$



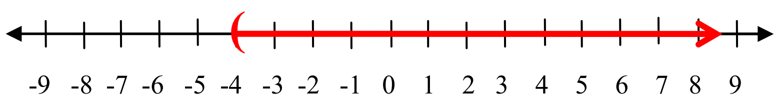
43. $\{x | -5 \leq x < 0\}$



45. $\{x | 1 \leq x \leq 8\}$



47. $\{x | x \geq -4\}$



49. $[-10, 5]$

50. $(-1, \infty)$

51. $(-6, 4]$

52. $(0, 3)$

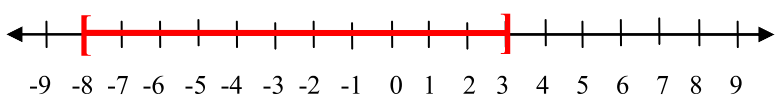
53. $(-\infty, -1]$

54. $(-\infty, 2]$

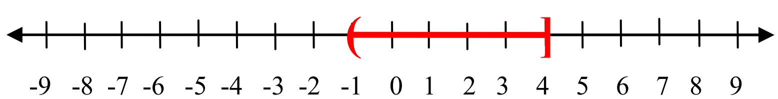
55. $(-2, \infty)$

56. $[-7, -4)$

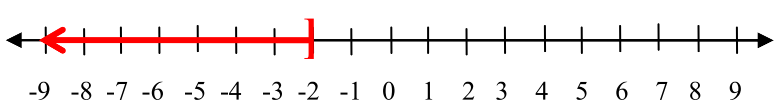
57. $\{x | -8 \leq x \leq 3\}$



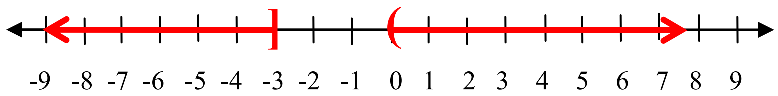
59. $\{x | -1 < x \leq 4\}$



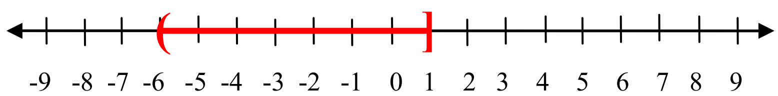
61. $\{x | x \leq -2\}$



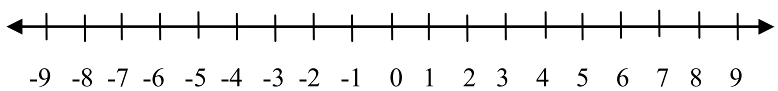
63. $\{x|x > 0\} \cup \{x|x \le -3\}$



65. $\{x|x > -6\} \cap \{x|x \le 1\}$



67. $\{x|x < -3\} \cap \{x|x > 4\}$



This intersection is empty